University of Padova

Wireless Comunications Project

Markov Fading Channels

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Abstract

In this report several Finite-State Markov Chain (FSMC) models found in literature, both in one and two dimension, for the received signal of a flat Rayleigh fading wireless channel are mathematically presented and then evaluated through computer simulations. Different approaches for the selection of the chain's parameters have been studied and are briefly explained: *equal-probability*, *equal-duration*, *adjacent-transition*, *linear-increasing probability*, *exponential-increasing probability* and *uniform partition* as 1D models and a 2D model with a custom optimization for the amplitudes and equiprobable partitioning for the rates. First and second order statistics and the computational time are analyzed. Furthermore a simple Selective Repeat protocol has been used to verify the accuracy of the Markov models on upper layers. Moreover some considerations about channel capacity were made and a new partitioning-method is proposed in order to maximize the capacity. Finally the overall conclusions are reported in order to shed light on the main advantages and limitations of the Markov modelings.

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1 Introduction

In typical urban environments the most used statistical model for the characterization of a radio signal's propagation is the Rayleigh fading. This type of model takes into account the random variations of the attenuation of the channel, referred as fade (fading produces small-scale effects that depend on the multipath propagation and there must be other models, such as path loss and shadowing, upon which it is superimposed) and these fluctuations are characterized by the Doppler frequency effect, which is due to the motion of the mobile terminal. For this reason Rayleigh fading represents a good model in case there is not a *Line of Sight* (LoS) between the transmitter and the receiver (otherwise Rician fading may be more suitable).
However, the complex mathematical characterization of fading carries out difficulties about analytical calculations of the channel's performances, such as the average packet error probability, that would probably have a non-closed form equation. Therefore, in order to develop upper layer procedures (e.g. *QoS* provisioning) some simpler models have been proposed. In the following the *Finite State Markov Channel* (FSMC) is discussed as one of them: its main idea is the quantization both in time and in amplitude of the fading process.

For their fundamental importance FSMC models for Rayleigh fading were deeply studied in the last two decades. Basically because the Markov model is characterized by more variables than equations in order to set the fading thresholds and the transition probabilities (as will be more evident later on), different design choices are possible and many studies have been done in that way so far.
For this reason a view of some selected modeling techniques is provided and the evolution of such a process is discussed in a chronological order, trying to point out the most innovative idea against the previous literature and the results are compared.

In addition we implemented a simple Data Link Layer protocol such as *Selective Repeat Automatic Repeat reQuest* (SR-ARQ) upon the FSMCs and we compared some performance metrics between the usage of these models instead of more complex simulators, such as the Jakes' one. The most important aspect is the pretty good accuracy of the Markov models with refer to e.g Jakes' simulator, but also their very low computational load.

Furthermore we analyzed the effect on the capacity obtained by the FSMCs and the theoretical value; these considerations lead to a new partitioning proposal.

The remainder of this report is organized as follows: Section 2 describes the overall objectives, scenarios and Markov channels considered. Section 3 provides the accuracy analysis of some selected papers and Section 4 draws the conclusions of the report.

2 Technical Approach

2.1 Objectives

In this report we want to provide a brief but also complete view of the most common uses of FSMC for the simulation of Rayleigh fading channels. The parameters that influenced the construction of the chain are identified and some FSMC's performances are evaluated in different configurations. Finally general pros and cons of this simplified models are analyzed.

2.2 Scenario

As already mentioned FSMCs are built in order to simplify the mathematical characterization of the Rayleigh wireless channels by quantizing both the time and the multipath envelope at the receiver. For these models to hold, the fundamental assumption of flat fading is made (i.e. all the frequency components of the transmitted signal are affected by the channel and its coherence bandwidth is larger than the bandwidth of the signal). Hence the channel effects can be modeled as $h(t) = h_{phase}(t) + j h_{quad}(t)$, where the phase and the quadrature components are independent and Gaussian distributed with mean zero and variance b_0. The envelope of the fading channel is computed as $r(t) = \sqrt{h^2_{phase}(t) + h^2_{quad}(t)}$ and can be proved to be Rayleigh distributed with mean $\sqrt{\frac{b_0 \pi}{2}}$.
Alternately, the instantaneous SNR $\gamma(t)$ at the receiver can be used as a good indicator of the state of the channel. The larger $\gamma(t)$ is, the "better" the channel conditions are. It is computed as $\gamma(t) = |h(t)|^2 \frac{E_s}{N_0}$, where E_s is the average energy per symbol and N_0 is the noise power spectral density. Under the assumption of Gaussian Noise, $\gamma(t)$ is exponentially distributed for Rayleigh fading channels.

2.3 Mathematical Models ans Equations

In this section is explained how a Markov Channel is built up given a Rayleigh fading distribution. For future convenience we want to model the SNR at the receiver, instead of the amplitude of the received signal envelope (in fact, given the SNR, it becomes easier the calculation of the average error probability in order to characterize the channel quality).
In presence of additive Gaussian Noise the Rayleigh fading SNR γ is exponentially distributed with pdf:

$$p(\gamma) = \frac{1}{\gamma_0} \exp\left(-\frac{\gamma}{\gamma_0}\right) \qquad \gamma \geq 0\,, \tag{1}$$

where γ_0 is the average SNR.

From the general Markov Chain theory a recall of some fundamental definitions that are needed to characterize the process is covered. Let $X = \{x_0, x_1, ..., x_{K-1}\}$ be the finite set of states and $\{X_n\}$ with $n \geq 0$ be the Markov process that identifies the channel state in the n^{th} time slot.

Then a K-states FSMC can be constructed by partitioning the range of the received SNR in K intervals and $K+1$ thresholds $\Gamma_0 < \Gamma_1 < ... < \Gamma_K$. Note that we generally set the first and the last thresholds as $\Gamma_0 = 0$ and $\Gamma_K = +\infty$ and the channel is said to be in state i if the SNR falls between Γ_i and Γ_{i+1}.

Thereafter, since we are interested in constant Markov processes, that have the property of stationary transitions, we can define the transition probabilities from state i to state j for a generic time index n as:

$$p_{i,j} = \mathrm{Prob}\left[X_{n+1} = x_j | X_n = x_i\right]. \tag{2}$$

In addition, the steady-state probabilities (i.e. the probability that the Markov process is in the i^{th} state at a certain time index n) can be defined as:

$$\pi_i = \mathrm{Prob}\left[X_n = x_i\right]. \tag{3}$$

Hence the i^{th} state is associated with a discrete SNR value SNR_i within the interval that this state represents. Generally this value is the statistical mean between the two thresholds:

$$\mathrm{SNR}_i = \frac{\int_{\Gamma_i}^{\Gamma_{i+1}} \gamma p(\gamma) d\gamma}{\pi_i} \qquad i = 0, ..., N-1 \tag{4}$$

Some almost obvious constraints respectively for outgoing flows sum, steady-state probabilities range and sum follow:

$$\text{outgoing flows sum property}: \quad \sum_{j=0}^{K-1} p_{i,j} = 1 \qquad \forall\, i \in \{0, 1, ..., K-1\}, \tag{5}$$

$$\text{steady}-\text{state range}: \quad 0 < \pi_i \leq 1 \qquad \forall\, i \in \{0, 1, ..., K-1\}, \tag{6}$$

$$\text{steady}-\text{state sum}: \quad \sum_{i=0}^{K-1} \pi_i = 1. \tag{7}$$

Steady-state and transition probabilities must also satisfy the equilibrium condition, or flow equation, which states that the incoming flow and the outgoing one must be equal for any given state k:

$$\sum_{j=0}^{K-1} \pi_j p_{j,k} = \sum_{i=0}^{K-1} \pi_k p_{k,i} \;\; \forall\, k \in \{0, 1, ..., K-1\}, \tag{8}$$

that can be easily simplified thanks to (5) as:

$$\sum_{j=0}^{K-1} \pi_j p_{j,k} = \pi_k \;\; \forall\, k \in \{0, 1, ..., K-1\}\,. \tag{9}$$

Finally, a complete description of a FSMC also requires information about the binary symmetric channel and the initial probabilities associated with each state.

In the following we only consider the binary symmetric channel associated with the generic i^{th} state as the *Bit Error Rate* (BER) indicated as e_i given by the modulation choice; of course, there are two quite self-evident constraints:

$$0 \leq e_i \leq 0.5\,, \qquad \forall\, k \in \{0, 1, ..., K-1\}\,, \tag{10}$$

$$e_i \neq e_j,\ \text{if}\ i \neq j \quad \forall i,\ j \in \{0, 1, ..., K-1\}\,; \tag{11}$$

where the eq. (11) declares that each state (i.e. SNR level) corresponds to a different BER.

Let now f_m be the maximum Doppler frequency determined by the motion of the mobile terminal and defined as:

$$f_m = \frac{v}{\lambda}\,, \tag{12}$$

where v is the speed of the mobile terminal and λ is the wavelength.

Then we can define the level crossing rate N of the SNR as the number of times per second the received SNR passes down across a given threshold Γ_{th} and we can prove it to be:

$$N(\Gamma_{th}) = \sqrt{\frac{2\pi\Gamma_{th}}{\gamma_0}} f_m \exp\left(-\frac{\Gamma_{th}}{\gamma_0}\right). \tag{13}$$

Concluding the general argumentation of the FSMC model we can observe that some parameters are left as design choice, such as the number of states and the way the received SNR has to be partitioned. In the following some ideas, found in literature, are briefly explained.

First of all, the model proposed by Moayeri and Wang in [1] is examined. The fundamental assumption of this model and of all the following is the slow fading condition, that "reasonably" leaded the authors to neglect transitions between two non-adjacent states, that means:

$$\begin{cases} f_m T_S \ll 1 \\ p_{i,j} = 0 \quad \forall\, |i-j| > 1 \end{cases} \tag{14}$$

where T_S is the slotted time interval (and for the sake of clarity, in the following we assume T_S as the time needed to transmit a packet T_P).

Here the simplest partitioning idea was performed, i.e. the SNR thresholds were determined by the *equal-probability method* such that the following condition holds:

$$\pi_1 = \pi_2 = \ldots = \pi_K = \frac{1}{K}, \tag{15}$$

whose rationale is that, in this way, no state is under-used. Hence, starting from (15) the following $K-1$ equations can be solved in an iterative manner from Γ_1 onwards:

$$\pi_i = \int_{\Gamma_i}^{\Gamma_{i+1}} p(\gamma) d\gamma = \frac{1}{K}\,. \tag{16}$$

Considering a communication system with transmission rate R_t symbols per second, there are on average $R_t^{(i)}$ symbols per second transmitted in the i^{th} state, with:

$$R_t^{(i)} = R_t \cdot \pi_i, \tag{17}$$

and therefore the transition probabilities have been approximated as:

$$p_{i,i+1} \approx \frac{N_{i+1}}{R_t^{(i)}} \qquad i = 0, 1, \ldots, K-2 \tag{18}$$

$$p_{i,i-1} \approx \frac{N_i}{R_t^{(i)}} \qquad i = 1, 2, \ldots, K-1 \tag{19}$$

$$p_{i,i} = 1 - p_{i,i-1} - p_{i,i+1} \qquad i = 1, 2, \ldots, K-2 \tag{20}$$

$$p_{0,0} = 1 - p_{0,1} \qquad\qquad p_{K-1,K-1} = 1 - p_{K-2,K-2} \tag{21}$$

hence, given the number of states, they built the FSMC model. As a first result it is easy to prove that the equilibrium equations hold for each state; in addition computer analysis were performed in order to validate the accuracy of this model as explained in Section 3.

This model was further improved by the *equal-duration method* proposed in [2] by Zhang and Kassam. Under the assumptions of adjacent-transition only (see eq. (14)) they proposed a FSMC where the average duration of the received SNR $\bar{\tau}_i$ between Γ_i and Γ_{i+1} is assumed to be a multiple of the packet transmission time T_P (which can be considered, of course, the inverse of R_t defined above), i.e.:

$$\bar{\tau}_i = c_i T_P. \tag{22}$$

Where $\bar{\tau}_i$ is computed as the ratio of the probability of being in state i over the number of time per second that the SNR passes across the thresholds Γ_i and Γ_{i+1}:

$$\bar{\tau}_i = \frac{\pi_i}{N(\Gamma_i) + N(\Gamma_{i+1})}. \tag{23}$$

Then, using (22) and (23) we can easily find K equations in $2K + 1$ variables (the c_i for $i = 0, \ldots, K-1$, the number of states K and Γ_i for $i = 1, \ldots, K-1$, since they had set $\Gamma_0 = 0$ and $\Gamma_K = \infty$). Now, requiring the c_i to be all equal to a constant c (so that each state has the same average time duration) we can solve for c and for the thresholds given each fixed K of interest. Of course, as $f_m T_P$ decreases so c decreases and conversely, because the more the channel is slowly fading, the more the time duration of each SNR state lasts.

Suddenly they proposed an interesting viewpoint in order to reduce of one unit the number of states needed and therefore to reduce the complexity of the model. They noticed that the last state (which represents very high SNR values) may be underused and so they truncated the SNR extension up to Γ_t; where it's defined as:

$$\text{Prob}[\gamma > \Gamma_t] = \exp(-\frac{\Gamma_t}{\gamma_0}) = \epsilon \tag{24}$$

in this way they can monitor this *outage probability* through ϵ (the used value is $\epsilon = 10^{-3}$). A similar approach have been used by many successive studies.

Then another interesting model was proposed by Park and Hwang in [3], where they implemented the so-called *adjacent-transition method*. Under the typical assumption of slow fading, initial thresholds set as the previous works ($\Gamma_0 = 0$ and $\Gamma_N = +\infty$) and transitions between adjacent states only (see eq. (14)), they realized a method that outperforms the previous two in many cases.

In order to describe their method they introduced the conditional pdf $p_{\gamma_r|\gamma}(\gamma_r|\gamma)$ that the received SNR value is γ_r in the next time interval given that at the current one it is γ as:

$$p_{\gamma_r|\gamma}(\gamma_r|\gamma) = \frac{1}{\left(1 - J_0^2(2\pi f_m T)\right)\gamma_0} I_0\left(\frac{2\sqrt{J_0^2(2\pi f_m T)\gamma\gamma_r}}{\left(1 - J_0^2(2\pi f_m T)\right)\gamma_0}\right)\exp\left(-\frac{J_0^2(2\pi f_m T)\gamma + \gamma_r}{\left(1 - J_0^2(2\pi f_m T)\right)\gamma_0}\right), \tag{25}$$

where J_0 an I_0 respectively are the Bessel function and the modified Bessel function of the first kind and zero order. Then they computed the probability of adjacent-transitions (from the state i to the states i, $i-1$ or $i+1$ only, in a single step) and non-adjacent ones as:

$$\begin{cases} P_{ad}(i) = \dfrac{1}{\pi_i}\displaystyle\int_{\Gamma_i}^{\Gamma_{i+1}}\int_{\Gamma_{i-1}}^{\Gamma i+2} p_{\gamma_r|\gamma}(\gamma_r|\gamma)p_\gamma(\gamma)d\gamma_r d\gamma \\ P_{non-ad}(i) = 1 - P_{ad}(i) \end{cases} \tag{26}$$

They want to keep $P_{non-ad}(i)$ sufficiently small for each state i (ideally equal to zero); therefore they proved that for a sufficiently small $\epsilon > 0$ such that $\int_{\Gamma_{i-1}}^{\Gamma_{i+1}} p_{\gamma_r|\gamma}(\gamma_r|\Gamma_i) d\gamma_r = 1 - \epsilon \quad i = 1, \ldots N-1$ it holds that $P_{non-ad}(i) \leq 2\epsilon$. Hence they chose Γ_{N-1} such that the outage probability (as in (24)) is 10^{-4} and they further proved that the $N-2$ remaining thresholds and the ϵ value can be found thanks to:

$$Q\left(\sqrt{\frac{2J_0^2(2\pi f_m T)\Gamma_i}{\left(J_0^2(2\pi f_m T)\right)\gamma_0}}, \sqrt{\frac{2\Gamma_{i-1}}{\left(J_0^2(2\pi f_m T)\right)\gamma_0}}\right) - Q\left(\sqrt{\frac{2J_0^2(2\pi f_m T)\Gamma_i}{\left(J_0^2(2\pi f_m T)\right)\gamma_0}}, \sqrt{\frac{2\Gamma_{i+1}}{\left(J_0^2(2\pi f_m T)\right)\gamma_0}}\right) = 1 - \epsilon \quad (27)$$

where $i = 1, \ldots, N-1$ and Q is the Marcum Q-function of order 0. The last check concerns if the ϵ value is *small enough*; otherwise the procedure must be repeated decreasing the number of states N (the more N increases the more the SNR intervals are narrow and the more ϵ is large).

Furthermore in [4] a review of many methods found in literature is proposed (in addition to the ones already discussed they studied the *linearly increasing probabilities* and the *exponentially increasing probabilities* methods) and they also explained their own new idea: the *uniform partition* method. The whole study assumes that the fading process is sufficiently slow and that only adjacent-transitions can occur (as in eq. 14).

In the *linearly increasing probabilities* the probabilities of the states are made linearly increasing with higher SNR values:

$$\pi_i = (i+1)\pi_0 \quad \text{with} \quad \pi_0 = \frac{2}{N^2 + N}, \quad i = 1, \ldots N. \quad (28)$$

In the *exponentially increasing probabilities* the probabilities of the states are made exponentially increasing with higher SNR values:

$$\pi_{i+1} = 2\pi_i \quad \text{with} \quad \pi_0 = \frac{1}{2^N - 1}, \quad i = 0, \ldots N-1. \quad (29)$$

The *uniform partition* method follows a sort of "reverse philosophy". First of all the last threshold is selected (similarly to eq. (24)), then the SNR thresholds are uniformly defined and the state probabilities are derived from them.

However, in order to achieve better performances recent studies have moved into *2D* FSMCs' analysis, because the multivariate Rayleigh fading distribution is also obviously intractable (these reasons will be clearer thanks to simulation results provided in the next section). In fact, also in multivariate problems the theoretical model is not feasible leading to difficult mathematical analysis or non-closed forms.

In these models each state represents both the amplitude and the rate-of-change of the fading process. Hence the state-space X is divided into n amplitude states and m rate states.

For $2D$ FSMC models we refer to [5] by Carruthers and Beaulieu. There they define the amplitude thresholds vector as $\delta = [\delta_0, ..., \delta_n]$ and the rate threshold vector as $\gamma = [\gamma_0, ..., \gamma_m]$ (as in previous works they set $\delta_0 = -\infty$, $\delta_n = +\infty$, $\gamma_0 = -\infty$, $\gamma_m = +\infty$). Each state represents a particular value of the fading amplitude and for a gaussian process is computed as:

$$f_G[d] = \frac{\sqrt{\frac{2b_0}{\pi}}\left(exp\left(-\frac{\delta^2_{i_d-1}}{2b_0}\right) - exp\left(-\frac{\delta^2_{i_d}}{2b_0}\right)\right)}{\text{Erf}\left[\frac{\delta_{i_d}}{\sqrt{2b_0}}\right] - \text{Erf}\left[\frac{\delta_{i_d-1}}{\sqrt{2b_0}}\right]} \tag{30}$$

for appropriate indexing i, where $d = 1, 2, ..., m \cdot n$ and $\text{Erf}[\cdot]$ is the error function.

The initial probability of being in state s_d is given by the vector:

$$\Phi_{0G}[d] = \frac{1}{4}\left(\text{Erf}\left[\frac{\delta_{i_d}}{\sqrt{2b_0}}\right] - \text{Erf}\left[\frac{\delta_{i_d-1}}{\sqrt{2b_0}}\right]\right) \cdot \left(\text{Erf}\left[\frac{\gamma_{j_d}}{\sqrt{2b_2}}\right] - \text{Erf}\left[\frac{\gamma_{j_d-1}}{\sqrt{2b_2}}\right]\right) \tag{31}$$

for appropriate indexing i and j and with $b_2 = 2\pi f_D^2 b_0$.

They also define $f_G(\mathbf{T_c})$ as the joint distribution of the amplitude and rate of two consecutive samples and the transition probability matrix is computed as:

$$P_{G_{hg}} = \frac{\int_{\gamma_{j_g-1}}^{\gamma_{j_g}} \int_{\delta_{i_g-1}}^{\delta_{i_g}} \int_{\gamma_{j_h-1}}^{\gamma_{j_g}} \int_{\delta_{i_h-1}}^{\delta_{i_h}} f_G(\mathbf{T_c}) d\mathbf{T_c}}{\Phi_0[h]}. \tag{32}$$

These calculations has to be evaluated through numerical integration techniques (for example the algorithm proposed by Genz [6]). Then they provide a set of expression in order to transform the previous Gaussian model (suitable for the quadrature and the in-phase component of a fading channel) into the Rayleigh model of the envelope associated, as follows:

$$\begin{cases} \Phi_{0R} = \Phi_{0G} \otimes \Phi_{0G} \\ P_R = P_G \otimes P_G \\ \mathbf{f_R} = \mathbf{f_G} \oplus \mathbf{f_G} \end{cases} \tag{33}$$

where $\otimes$ is the Kronecker product and $\oplus$ is a variation on the Kronecker products (see eq. (15) in [5]).

For the construction of the chain, they chose a steepest descent algorithm with the inequality constraints enforced by a log barrier for the selection of the amplitude partitions in order to maximize the autocorrelation of the Gaussian fading at lag 0. In fact this can actually be formulated as an optimization problem. We used an heuristic approach adopting the *simulated annealing* algorithm that generally converges in a few steps.

Moreover they considered an equiprobable partitioning of the rate-space, i.e. by solving:

$$\frac{1}{2}\left(\mathrm{Erf}\left[\frac{\gamma_d}{\sqrt{2b_2}}\right]-\mathrm{Erf}\left[\frac{\gamma_{d-1}}{\sqrt{2b_2}}\right]\right)=\frac{1}{m}. \tag{34}$$

2.4 Complications found

The implementation of several techniques studied in literature brings out some programming difficulties. First of all we dealt with the inversion of the Marcum Q-function in MATLAB in order to find the SNR thresholds; we overtook this problem by learning to use Mathematica. Then we had some troubles in coding multiple-integral equations and in the optimization algorithm in order to realize the $2D$ FSMC model; a deeper learning of mathematical methods was necessary. These and other calculations pointed out that MATLAB is not always very *flexible*.

3 Results

In this section some numerical evaluations of the various FSMC models are presented. First and second order statistics and other performances parameter are examined through computer simulations in order to clearly show advantages and limitations of the FSMC modeling.

The most interesting parameters of the process in this report are the mean, the standard deviation, the AutoCorrelation Function (ACF) and the computation time. Moreover, a simple Selective Repeat (SR) ARQ protocol has been used to show once again the validity of the models (a similar study was proposed by [7] for a Go back N protocol) and some capacity considerations have been done.

We implemented all the FSMCs in MATLAB or Mathematica; the limitations of the one have been overtaken thanks to the other. In particular we implement in Mathematica the following procedures that MATLAB was not able to deal with: the computation of the SNR thresholds for the equal-duration and adjacent-transition methods (respectively because MATLAB lamented precision errors for more than 14 states and does not provide an useful inverse Q-function), the computation of the thresholds of amplitudes and rates for the $2D$ model (indeed we realized that Mathematica was probably more suitable for optimization problems as we did).

Some further definitions and other Rayleigh channel models are needed in order to properly understand the following analysis.

The ACF is defined as:

$$R[m]=E[\mathrm{x}(x_0)\mathrm{x}(x_m)]=\sum_{j=1}^{n}\mathrm{x}(j)\pi_j\sum_{i=1}^{n}\mathrm{x}(i)p_{i,j}^{m} \tag{35}$$

where $\mathrm{x}(\cdot)$ is the output of the Markov chain at the instant m. The theoretical Gaussian fading ACF from

	Mean ($K = 10$)	STD ($K = 10$)	Mean ($K = 20$)	STD ($K = 20$)
Equal-probability	1.2578	0.6432	1.2472	0.6508
Equal-duration	1.2619	0.6386	1.2572	0.6492
Adjacent-transition	1.2556	0.6422	1.2519	0.6478
Linearly-increasing probability	1.2646	0.6412	1.2457	0.6497
Exponentially-increasing probability	1.2847	0.5860	/	/
Uniform-partition	1.2553	0.6483	1.2530	0.6532
Theoretical	1.2533	0.6551	1.2533	0.6551

Table 1: Mean and Standard Deviation of the envelope of $1D$ FSMCs for slow-fading $f_m T_S = 10^{-2}$.

Clarke's model [8] (the "classical" model in case of isotropic scattering, i.e. identical power comes from each angle, that is a good assumption in case of many scatterers) is computed as [5]:

$$R_G(\tau) = b_0 J_0(2\pi f_m T_S \tau), \tag{36}$$

where τ is the time difference (i.e. the *lag*). For the respective fading envelope it holds [4]

$$R_r(\tau) = \frac{\pi b_0}{2} {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; (J_0(2\pi f_m \tau))\right), \tag{37}$$

where ${}_pF_q(\mathbf{n}, \mathbf{d}, z)$ is the hypergeometric function of of p and q. As regards the other methods for Rayleigh channel generation, we selected the following ones from the literature:

- the Jakes' simulator [9];
- the *Sum-of-Sinusoids* (*SoS*) method proposed by Zheng and Xiao in [10] with 15 sinusoids.
- the *SoS* method proposed by Clark with 15 sinusoids [8].

We begin the analysis with the $1D$ first order statistics of the fading envelope's mean and standard deviations. Results were measured with 10^7 samples, $b_0 = 1$, fading speed $f_m T_S = 10^{-2}$ and number of states $K = 10$ and 20. Table 1 shows a comparison between the values obtained through FSMC models and the theoretical values. All the methods, but the exponential with 20 states, can faithfully approximate both mean and standard deviation; we notice that the uniform is the one that performs slightly better at first order statistics in the considered situations. The exponential method is not taken into account for 20 states because the first thresholds are too small and close one to another, hence the adjacent transitions only assumption is no more verified.

The mean and the standard deviation of the $2D$ model, with $n = 10$ and $m = 4$ states (because, as we are going to see soon, this is a well-working configuration), are shown in Table 2. Also the $2D$ FSMC is able

	Mean	STD
$2D$ ($n = 10$, $m = 4$)	1.2593	0.6325
Theoretical	1.2533	0.6551

Table 2: Mean and Standard Deviation of the envelope for slow-fading $f_m T_S = 10^{-2}$.

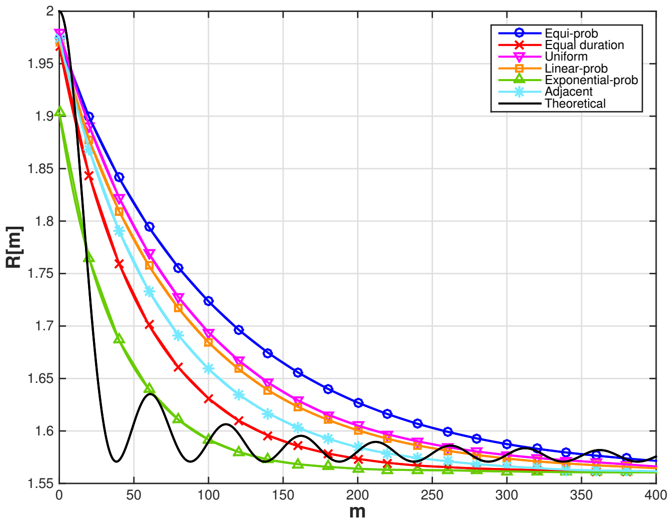

Figure 1: $1D$ ACFs of the fading envelope, 10 states, $f_m T_S = 10^{-2}$.

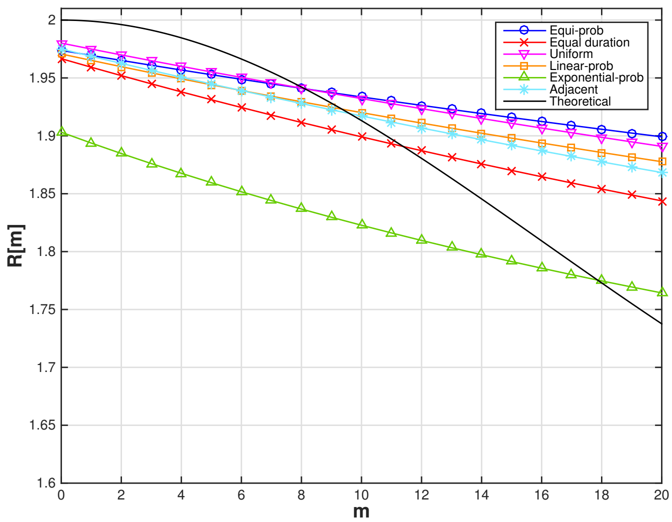

Figure 2: Zoom of Fig. 1.

to approach both first and second order statistics.

Moreover we tried to consider a medium-fading scenario with $f_m T_S = 10^{-1}$; that forces some logical errors and contradictions on the assumptions we made of adjacent-transition only. As expected we saw that the $1D$ FMSCs are not suitable to correctly model the channel, leading to strange and inaccurate results (not reported here because almost irrelevant). This motivates the introduction of the $2D$ FSMC models.

In order to further validate the $1D$ FSMC models we computed the ACFs for the fading envelope. For example in Fig. 1 the ACFs for the $1D$ models with $K = 10$ states and $f_m T_S = 10^{-2}$ are plotted and compared to the theoretical one obtained through eq. (37). We can see that the ACFs for the $1D$ Markov models decay exponentially in contrast to the theoretical Bessel function: this is one of the main limitations of this approach. In Fig. 2 we zoomed the Fig. 1 in the first samples of observation because these are the only ones that the FSMCs can properly approximate. Hence we can see that the uniform-method is the one that performs better, confirming first order statistics analysis. A different configuration is analyzed in Fig. 3, here $K = 20$ states are considered. The respective zoom is shown in Fig. 4. We can see that, again, the uniform slightly outperform the others. We can also notice that an increasing number of states improve the similarity of the ACFs with the theoretical one only in the first samples; then the curves worsened. This is because that, with more states, the assumption of adjacent transitions is less verified. Finally we choose the best $1D$ model (i.e. the uniform partition method) and we studied how the variations on the number of

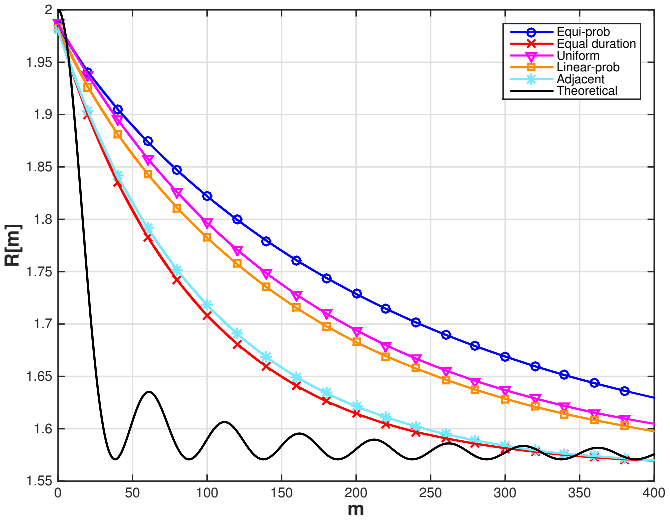

Figure 3: $1D$ ACFs of the fading envelope, 20 states, $f_m T_S = 10^{-2}$.

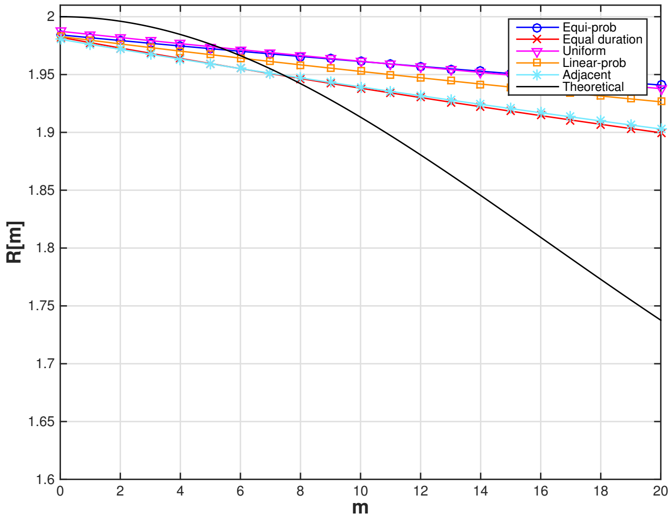

Figure 4: Zoom of Fig. 3.

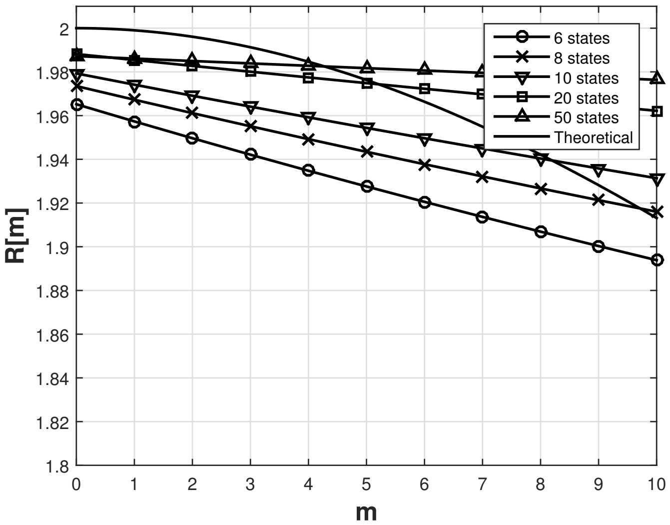

Figure 5: ACFs of the uniform for different number of states, $f_m T_S = 10^{-2}$.

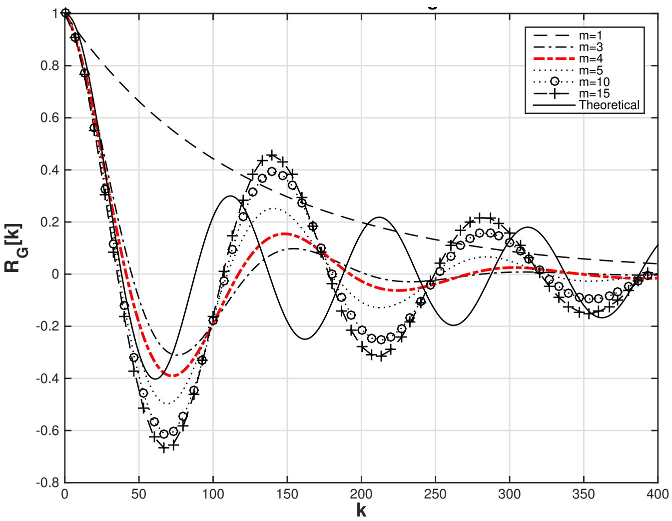

Figure 6: ACFs of the Gaussian envelope of the $2D$ model, $f_m T_S = 10^{-2}$ and $n = 10$.

states influences the model's accuracy. The result is plotted in Fig. 5. Intuitively, an higher number of states correspond to a better approximation of the theoretical ACF, but, again, if the number of states increase too much the assumption of adjacent-transition only it will not be true anymore.

As regards $2D$ FSMC, we analyzed its ACF for different settings. In Fig. 6 the ACF of the Gaussian fading of the $2D$ model is plotted varying the rate-state number (m) together with the theoretical one (computed as (36)) for fixed number of amplitude-states $n = 10$ and $f_m T_S = 10^{-2}$. Here we can observe that for $m = 1$ there is an exponential decay, because it is actually a $1D$ model. Therefore we can notice that the more the rate-states are, the more the magnitude of oscillations increases; however no significant effects are provided in terms of oscillation frequency and decay rate. Hence this procedure it is not very effective and three or four rate-states are sufficient (this is intuitively confirmed because of the slow-fading assumptions,

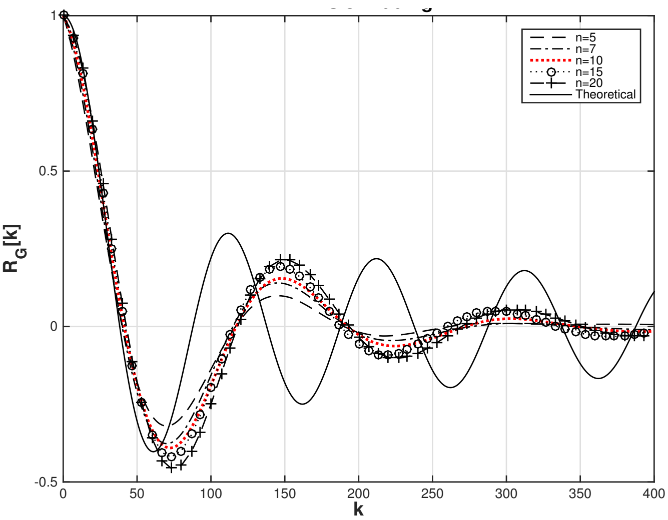

Figure 7: ACF of the Gaussian envelope of the $2D$ model, $f_m T_S = 10^{-2}$ and $m = 4$.

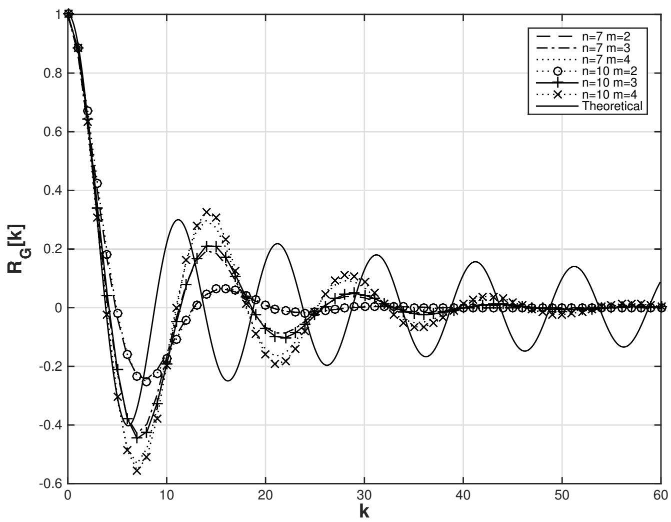

Figure 8: ACF of the Gaussian envelope of the $2D$ model with medium-fading $f_m T_S = 10^{-1}$.

thus the rate is not supposed to be very heterogeneous); from the plot we can see that $m = 4$ (the red line in Fig. 6) is the best setting in terms of approximation of the theoretical ACF in the first samples.
For this reason we tried to change the number of amplitude-states (n) fixing $m = 4$ rate-states in the same fading-conditions as before. The results are shown in Fig. 7; we can observe that in these configurations slight changes occur only in the magnitude of the oscillations; as before, no significant effects on the oscillation frequency and the decay rate happen.
Then we were interested in evaluating the performances for medium-fading channels; hence we considered a few well-working configurations from above and we plotted their ACFs for $f_m T_S = 10^{-1}$. The result is shown in Fig. 8 where we can observe a quite good behavior (immensely better than the one that can be obtained for $1D$ models).

Furthermore we analyzed the computational demand, in terms of elapsed time, requested by the FSMCs and we made the comparison with other channel generators. Table 3 shows the results we obtained to generate 10^6 complex samples. Looking at the computational load, the $1D$ FSMCs have the strong assumptions that only transitions to adjacent states are possible. This means that at each clock we have $O(1)$ operations thus making the complexity $O(t)$, where t is the sample number to generate. On the other hand, in transitions for $2D$ FSMCs, the former assumption is relaxed so that, in general, one state can evolve into any other state with a certain probability. Being the number of states $m \cdot n$, we have that at each clock the number of operations is $O(m \cdot n)$, hence the complete algorithm becomes $O(t \cdot m \cdot n)$. We can see that the $1D$ models perform much better than all the other methods, as regards computational time. $2D$ models lead to an increasing computational load in order to have better second order statistics performances.

Moreover, the impact of the FSMC modeling on upper layers is shown by the performances evaluation of a simple *SR-ARQ* protocol. In particular we measured the average per packet delay and the normalized throughput, computed as the ratio between the total number of packets and the total number of time slot needed (in fact we considered that a time slot is the time needed to sent one packet), with refer to the Jakes' simulator. The results are plotted, with zoom, in Fig. 9 and 10 for 10^7 packets of 128 bytes each, with fading speed $f_m T_S = 10^{-2}$ using *BPSK* modulation. From these figures we can see, as expected, that the $2D$ FSMCs perform better than the $1D$ models as approximations of the Jakes' model. The best $1D$ method is, again, the uniform-partition method.

Method	Time (s)
Equal-probability	0.0844
Equal-duration	0.0665
Adjacent-transition	0.0663
Linearly-incr. prob.	0.0697
Exponentially-incr. prob.	0.0700
Uniform-partition	0.0682
$2D$ ($n = 10$, $m = 4$)	5.2724
Jakes' Simulator	12.0284
SoS by Zheng and Xiao	0.5099
SoS by Clark	0.6121

Table 3: Computational time to generate 10^6 complex samples. $1D$ considered $K = 10$ states.

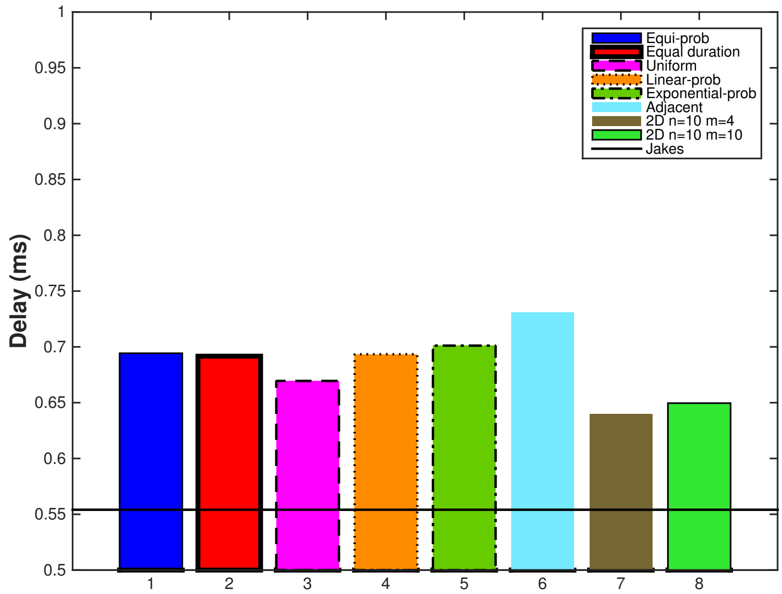

Figure 9: Per-packet average delay of *SR-ARQ*.

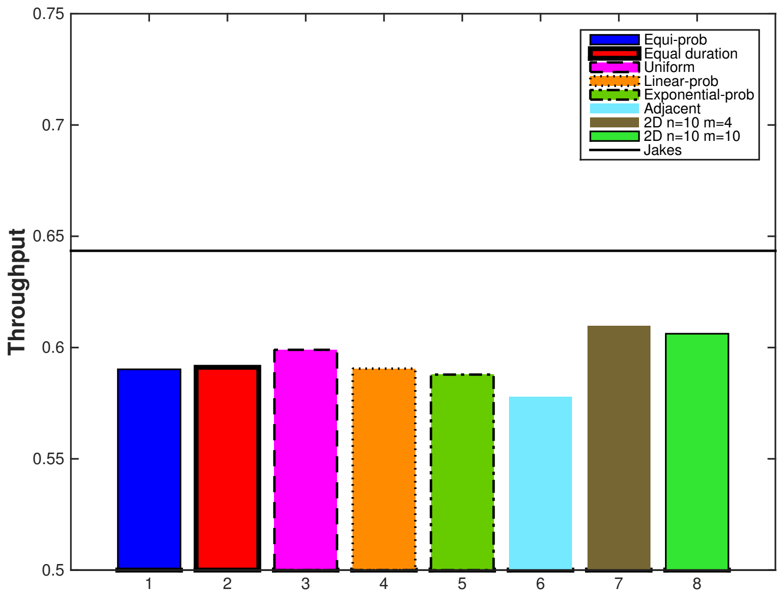

Figure 10: Normalized throughput of *SR-ARQ*.

Method	η (bit/s/Hz)
Equal-probability	2.0235
Equal-duration	2.0347
Adjacent-transition	2.0277
Linearly-incr. prob.	2.0280
Exponentially-incr. prob.	2.0820
Uniform-partition	2.0240
$2D$ ($n = 10$, $m = 4$)	2.0062
Theoretical	2.9065

Table 4: Spectral efficiency for $f_m T_s = 10^{-2}$.

From these considerations (with slow fading $f_m T_S = 10^{-2}$) is evident that the $2D$ FSMCs are *de facto* better for upper layer calculations but this came at a price of much more complexity; indeed, in many cases, it is maybe better to use the very lightweight $1D$ models. Otherwise, in medium fading conditions (e.g. $f_m T_S = 10^{-1}$) the $1D$ FSMCs do not work anymore but $2D$ models still do their job pretty well (as already seen).

Finally we analyzed the similarity between the spectral efficiency of the FSMCs and the theoretical, computed as

$$\eta = \int_0^{+\infty} \log_2(1+\gamma) p_\Gamma(\gamma) d\gamma. \tag{38}$$

Table 4 shows this comparison: here we can see that the $1D$ FSMCs perform slightly better than the $2D$ model. This arises a new partitioning paradigm based on the attempt to optimize the spectral efficiency. In fact it can be formulated as a mathematical problem and from the optimization of η can be derived the thresholds. Although it may seem quite easy, the math beyond that is not very simple; so we kept it as a proposal for future work. We think that this approach could work but it is actually too difficult to say in advance; the goodness of this idea has to be verified through the other performance metrics shown previously in this report (e.g. mean, standard deviation, ACF).

4 Conclusions and Future Works

Thanks to this work we improved MATLAB coding and its symbolic language; we also learned the basics of Mathematica, a very powerful software. We built up different channel simulators and we learned to compare their performances.

In this report we wanted to show the advantages and the limitations of the FSMCs. The convenient aspects basically are the high computational efficiency with refer to the continuous time simulators (ref. Table 3) and the good approximation of first order statistics (ref. Tables 1 and 2). The main disadvantage, instead, is the exponential decay of the $1D$ models' ACFs (aspect partly overtaken by the introduction of the $2D$ FSMCs at the cost of more complexity) in contrast to the theoretical one, which is oscillating.

These considerations allow us to conclude that the use of the $1D$ FSMC is justified when the applications require only the first coefficient of the ACF and the fading is slow; otherwise it may be preferable to use the $2D$ model or the theoretical one.

As a future work it might be interesting to compare more characteristics between the FSMCs and other channel simulators. For example it could be interesting to compute the power margin quality measures in order to further evaluate the validity of the FSMC models with refer to the theoretical one.

It could be also useful to study other channel modelings and other FSMC models present in literature and compare with the ones analyzed in this report in an holistic view.

Finally we would like to implement our proposal on capacity optimization and evaluate the respective performances.

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